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**Is It Possible To Embed A
4D, $\mathcal{N} = 4$ Supersymmetric Vector Multiplet
Within A Completely Off-Shell Adinkra Hologram?**

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ABSTRACT

We present evidence of the existence of a 1D, $N = 16$ SUSY hologram that can be used to understand representation theory aspects of a 4D, $\mathcal{N} = 4$ supersymmetrical vector multiplet. In this context, the long-standing “off-shell SUSY” problem for the 4D, $\mathcal{N} = 4$ Maxwell supermultiplet is precisely formulated as a problem in linear algebra.

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1 Introduction

The utility of the extra dimension concept underwent a reassessment due to the construction of 11D supergravity [1, 2]. Since then, the concept has received much attention even to the point of producing a broadly studied approach [3, 4, 5] (e. g. ‘brane-world scenarios’) that dominated phenomenological discussion for a decade. The 11D approach was the solution to a difficult problem as Cremmer and Julia used it to present the first complete description of 4D, $\mathcal{N} = 8$ supergravity [6].

Thus, for perhaps the first time in the literature associated with supergravity, it was shown that a higher dimensional approach contained information about a lower dimensional theory that could be more easily accessed from the higher dimensional starting point. This same idea can be seen in the relation of 10D, $\mathcal{N} = 1$ SUSY YM theories to 4D, $\mathcal{N} = 4$ SUSY YM theories. The key point to note is that the information necessary for the construction of both the higher dimensional and lower dimensional theories is conserved by either the dimensional reduction or dimensional extension processes. The lower dimensional theory acts as a hologram for the higher dimensional one. These facts are well known.

Starting in 1994 [7], we began to find evidence [8, 9, 10, 11] that this well known result extends all the way from supersymmetric quantum field theories to supersymmetric quantum mechanical models and more unexpectedly the conservation of the information may be so robust that it might allow the former to be re-constructed from the latter in some limits. Eventually we gave this idea a name “SUSY holography” [12, 13].

The topic of the 4D, $\mathcal{N} = 4$ Yang-Mills supermultiplet [14, 15, 16] has been a fruitful one for many years. Almost from the instant of its first presentation, the unusual properties of this model have generated a steady stream of inspirations concluding most recently with the introduction of the “amplituhedron” [17, 18]. Thus, this theory has long been one of our objectives to study via the tools that have been developed for SUSY holography.

Stated another way, it is our goal to follow a path similar to that of Cremmer and Julia, but to use the idea of “SUSY holography” in the reverse route of using a lower dimensional construct, at least at the level of representation theory, to gain greater understanding of a higher dimensional construct. One of our previous works [19], presented (what may be the most) detailed results on the nature of the non-closure of the SUSY algebra for the 4D, $\mathcal{N} = 4$ supermultiplet in the context of an equivalent $\mathcal{N} = 1$ superfield formulation solely in four dimensions. In this current

work, we will begin the process of studying its projection (or shadow) into the sea of one dimensional $N = 16$ adinkra networks known to exist.

2 The SUSY Holography Conjecture

On first reflection, the proposal of “SUSY holography” would seem untenable. There is an easy example to show why this conclusion might be reached. For a four dimensional field theory (supersymmetrical or not), the starting point for a reduction to one dimension can be implemented by making the replacement

$$\partial_\mu = \mathcal{T}_\mu \frac{\partial}{\partial \tau} \quad , \quad \mathcal{T}_\mu \equiv (1, 0, 0, 0) \quad , \quad (2.1)$$

in actions. As well, all field variables are assumed to depend only on the real parameter τ and all gauge fields are restricted to the Coulomb gauge. In the context of a non-supersymmetrical theory, such a reduction can lead to an ambiguity involving a loss of spin-bundle information. Let us consider two distinct four dimensional theories:

- (a.) an action involving three parity-even massless spin-0 fields $\phi^\mathcal{I}$, (where $\mathcal{I} = 1, 2$, and 3)

$$\mathcal{L}_{Spin-0} = -\frac{1}{2}(\partial_\mu \phi^\mathcal{I})(\partial^\mu \phi^\mathcal{I}) \quad , \quad (2.2)$$

so that under the prescription of (2.1) leads to

$$\mathcal{L}_{Spin-0} = \frac{1}{2}(\partial_\tau \phi^1)(\partial_\tau \phi^1) + \frac{1}{2}(\partial_\tau \phi^2)(\partial_\tau \phi^2) + \frac{1}{2}(\partial_\tau \phi^3)(\partial_\tau \phi^3) \quad , \quad \text{and} \quad (2.3)$$

- (b.) an action for a spin-1 gauge field given by

$$\mathcal{L}_{Spin-1} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}F_{0i}F^{0i} \quad . \quad (2.4)$$

(since all spatial derivatives vanish in our reduction scheme) and following the prescription above this becomes,

$$\mathcal{L}_{Spin-1} = \frac{1}{2} \left[(\partial_\tau A_1)^2 + (\partial_\tau A_2)^2 + (\partial_\tau A_3)^2 \right] \quad . \quad (2.5)$$

As is seen above, the forms of (2.3) and (2.5) are exactly the same. Thus, starting from a non-supersymmetrical one dimensional theory involving three bosonic fields there is no way to distinguish which of the four dimensional actions were its origin. Specifically, the information on the 4D spin-bundle of the fields was lost. This is an example of what we refer to as loss of information under non-supersymmetric 0-brane reduction described by (2.1).

Remarkably, within the context of supersymmetrical theories, this information can be conserved...if one looks into the “correct” structure.

While it is true that the information about the 4D origins of the actions does not appear in either 1D action, if these non-supersymmetrical theories are embedded within 1D, $N = 4$ theories at one extreme and 4D, $\mathcal{N} = 1$ theories on the other, the information can be subtly encoded in the SUSY variations!

We begin with the spin-0 field and for the sake of simplicity, we only need to consider a single such field. Since it has parity-even, it becomes the A -field in a chiral supermultiplet as part of the collection of fields (A, B, ψ_a, F, G) that appears in the set of equations

$$\begin{aligned} D_a A &= \psi_a \quad , \quad D_a B = i(\gamma^5)_a{}^b \psi_b \quad , \\ D_a \psi_b &= i(\gamma^\mu)_{ab} (\partial_\mu A) - (\gamma^5 \gamma^\mu)_{ab} (\partial_\mu B) - i C_{ab} F + (\gamma^5)_{ab} G \quad , \\ D_a F &= (\gamma^\mu)_a{}^b (\partial_\mu \psi_b) \quad , \quad D_a G = i(\gamma^5 \gamma^\mu)_a{}^b (\partial_\mu \psi_b) \quad , \end{aligned} \quad (2.6)$$

which imply the supersymmetry property

$$\{ D_a, D_b \} = i 2 (\gamma^\mu)_{ab} \partial_\mu \quad . \quad (2.7)$$

Finally, the action given by

$$\mathcal{L}_{CM} = -\frac{1}{2} \partial_\mu A \partial^\mu A - \frac{1}{2} \partial_\mu B \partial^\mu B + i \frac{1}{2} (\gamma^\mu)^{bc} \psi_b \partial_\mu \psi_c + \frac{1}{2} F^2 + \frac{1}{2} G^2 \quad , \quad (2.8)$$

possesses a symmetry (up to a surface term) under the variations implied by (2.6).

In a similar manner, the spatial vector \vec{A} can be combined with its temporal component A_0 to form a 4-vector A_μ and becomes the gauge field in a vector supermultiplet among the collection of fields (A_μ, λ_a, d) that appears in the set of equations

$$\begin{aligned} D_a A_\mu &= (\gamma_\mu)_a{}^b \lambda_b \quad , \\ D_a \lambda_b &= -i \frac{1}{4} ([\gamma^\mu, \gamma^\nu])_{ab} (\partial_\mu A_\nu - \partial_\nu A_\mu) + (\gamma^5)_{ab} d \quad , \\ D_a d &= i(\gamma^5 \gamma^\mu)_a{}^b (\partial_\mu \lambda_b) \quad . \end{aligned} \quad (2.9)$$

Up to a gauge transformation on the spin-1 field these also satisfy the algebra described by (2.7). The action given by

$$\mathcal{L}_{VM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \frac{1}{2} (\gamma^\mu)^{bc} \lambda_b \partial_\mu \lambda_c + \frac{1}{2} d^2 \quad , \quad (2.10)$$

possesses a symmetry (up to a surface term) under the variations implied by (2.9).

A valise formulation that is off-shell is one where a set of bosonic variables $\Phi_i(\tau)$ and fermionic variables $\Psi_{\hat{k}}(\tau)$ *locally* satisfy the following realization under the action of a set of supercovariant derivatives D_I

$$D_I \Phi_i = i (L_I)_{i\hat{k}} \Psi_{\hat{k}} \quad \text{and} \quad D_I \Psi_{\hat{k}} = (R_I)_{\hat{k}i} (\partial_\tau \Phi_i) \quad , \quad (2.11)$$

with L-matrices and R-matrices satisfying

$$\begin{aligned} (L_I)_{i\hat{j}} (R_J)_{\hat{j}}^k + (L_J)_{i\hat{j}} (R_I)_{\hat{j}}^k &= 2 \delta_{IJ} \delta_i^k \quad , \\ (R_J)_{i\hat{j}} (L_I)_{\hat{j}}^{\hat{k}} + (R_I)_{i\hat{j}} (L_J)_{\hat{j}}^{\hat{k}} &= 2 \delta_{IJ} \delta_i^{\hat{k}} \quad . \end{aligned} \quad (2.12)$$

$$(R_I)_{\hat{j}}^k \delta_{ik} = (L_I)_{i\hat{k}} \delta_{\hat{j}\hat{k}} \quad , \quad (2.13)$$

and where the indices range as $I, J, \text{etc.} = 1, \dots, N$; $i, j, \text{etc.} = 1, \dots, d$; and $\hat{i}, \hat{j}, \text{etc.} = 1, \dots, d$ for integers d , and N .

Implementation of the reduction process described at the beginning of this section is not sufficient to arrive at a valise formulation of these 4D, $\mathcal{N} = 1$ supermultiplets. In order to obtain a valise will also require that we make the ‘field redefinitions’

$$F^{\mathcal{I}} \rightarrow \partial_\tau F^{\mathcal{I}} \quad , \quad G^{\mathcal{I}} \rightarrow \partial_\tau G^{\mathcal{I}} \quad , \quad d \rightarrow \partial_\tau d \quad , \quad (2.14)$$

after the reduction. In a subsequent chapter, we will obtain the valise formulation of the 4D, $\mathcal{N} = 4$ theory as the main new result of this work.

Applying all of this machinery to the components of a chiral supermultiplet we find

$$\begin{aligned} D_a A &= \psi_a \quad , \quad D_a B = i (\gamma^5)_a^b \psi_b \quad , \\ D_a F &= (\gamma \cdot \mathcal{T})_a^b \psi_b \quad , \quad D_a G = i (\gamma^5 \gamma \cdot \mathcal{T})_a^b \psi_b \quad , \\ D_a \psi_b &= i (\gamma \cdot \mathcal{T})_{ab} (\partial_\tau A) - (\gamma^5 \gamma \cdot \mathcal{T})_{ab} (\partial_\tau B) - i C_{ab} (\partial_\tau F) + (\gamma^5)_{ab} (\partial_\tau G) \quad , \end{aligned} \quad (2.15)$$

and in a similar manner for the components of the vector supermultiplet, one is led to

$$\begin{aligned} D_a A_i &= (\gamma_i)_a^b \lambda_b \quad , \quad D_a d = i (\gamma^5 \gamma \cdot \mathcal{T})_a^b \lambda_b \quad , \\ D_a \lambda_b &= -\frac{i}{2} ([\gamma \cdot \mathcal{T} \quad , \quad \gamma^i]_{ab} (\partial_\tau A_i) + (\gamma^5)_{ab} (\partial_\tau d) \quad . \end{aligned} \quad (2.16)$$

Both of the equations in (2.15) and (2.16) have exactly the form of (2.11).

Under the reduction described above, there is a way to begin solely with a one dimensional supersymmetrical theory as shown in (2.3) or (2.5) and determine which of the two four-dimensional theories could provide the starting point. The way this is done is to note that for a one dimensional supersymmetrical theory, with at least

four worldline SUSY charges, any action is also accompanied by an associated set of ‘L-matrices’ and ‘R-matrices’ [20] as defined in (2.11)

All three chiral supermultiplets will have the same set of L-matrices and R-matrices and as first derived in [20] these can take the forms

$$\begin{aligned}
(L_1)_{i\hat{k}} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, & (L_2)_{i\hat{k}} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \\
(L_3)_{i\hat{k}} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, & (L_4)_{i\hat{k}} &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \quad (2.17)
\end{aligned}$$

and

$$\begin{aligned}
(R_1)_{i\hat{k}} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{bmatrix}, & (R_2)_{i\hat{k}} &= \begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \\
(R_3)_{i\hat{k}} &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, & (R_4)_{i\hat{k}} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \quad (2.18)
\end{aligned}$$

The vector supermultiplet has the set of L-matrices and R-matrices as first derived in [20] to be

$$\begin{aligned}
(L_1)_{i\hat{k}} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, & (L_2)_{i\hat{k}} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \\
(L_3)_{i\hat{k}} &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, & (L_4)_{i\hat{k}} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad (2.19)
\end{aligned}$$

and

$$(R_1)_{i\hat{k}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad (R_2)_{i\hat{k}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$(\mathbf{R}_3)_{i\hat{k}} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad (\mathbf{R}_4)_{i\hat{k}} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}. \quad (2.20)$$

The work of [21] noted each L-matrix and R-matrix can be expressed in terms of a ‘Boolean factor’ denoted by $(\mathcal{S}^{(I)})_{i\hat{\ell}}$ which appears via

$$(\mathbf{L}_I)_{i\hat{k}} = (\mathcal{S}^{(I)})_{i\hat{\ell}} (\mathcal{P}_{(I)})_{\hat{\ell}\hat{k}}, \quad \text{for each fixed } I = 1, 2, \dots, N. \quad (2.21)$$

$$(\mathcal{S}^{(I)})_{i\hat{\ell}} = \begin{bmatrix} (-1)^{b_1} & 0 & 0 & \dots \\ 0 & (-1)^{b_2} & 0 & \dots \\ 0 & 0 & (-1)^{b_3} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \leftrightarrow \left(\mathcal{R}_I = \sum_{i=1}^d b_i 2^{i-1} \right)_b \quad (2.22)$$

(a diagonal matrix with real entries that squares to the identity) times an element of the permutation group $(\mathcal{P}_{(I)})_{\hat{\ell}\hat{k}}$. The matrices above can be associated with a class of topological objects given the name of “adinkras” [22] which are graphs that capture (with complete fidelity) the information in the matrices and nodal heights. In fact, if the adinkras are regarded as graphs or networks, the permutation factor within the L-matrices and R-matrices are the ‘adjacency matrices’ from graph theory.

The adinkras associated with the chiral and the vector supermultiplets, respectively, are shown below.

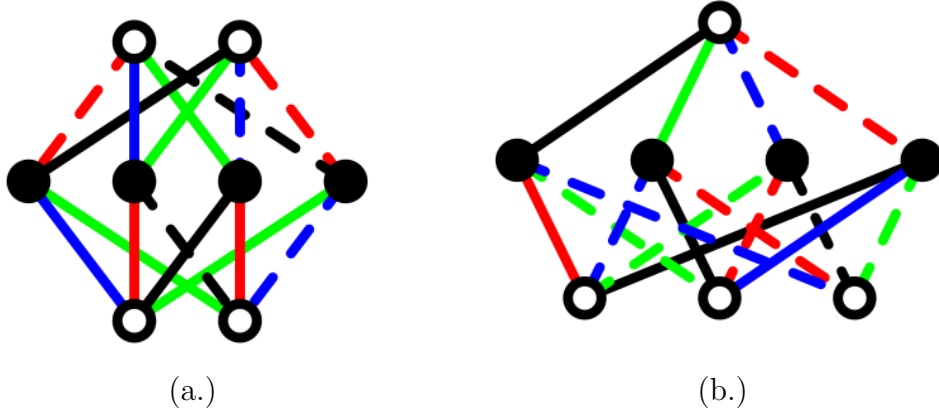


Figure # 1: Adinkra graphs for the chiral (a.) and vector (b.) supermultiplets

These are the graphs associated solely with the equations that appear in (2.6) and (2.9). On the other hand, the adinkra graphs associated with the valise equations of (2.15) and (2.16) are different and given in Figure # 2.

Written solely in the form of valise adinkras or their associated matrices in (2.19)-(2.20), it is not at all clear how the spin-bundle information to distinguish the chiral

supermultiplet from the vector supermultiplet has been retained. The question becomes, “What structure in the graphs or their associated matrices holographically stores the information about the distinction?”

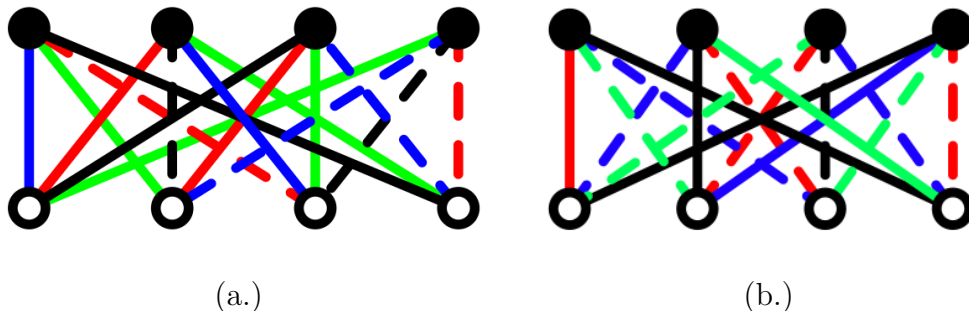


Figure # 2: Adinkra graphs for the valise chiral (a.) and valise vector (b.) supermultiplets

3 Adinkra Matrices and Information Conservation

The work of [21] has offered a proposal for identifying such a mechanism: the information may be accessed via the elements of the permutation group. To see most transparently how the permutation group elements contain the information we are seeking, it is useful to describe these elements in terms of cycles⁵.

To show this approach clearly, we will give an explicit demonstration using the L_1 matrix of the chiral multiplet.

$$\begin{aligned}
 (L_1)_{i\hat{k}} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\
 (L_1)_{i\hat{k}} &= (10)_b \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{3.1}
 \end{aligned}$$

where the Boolean factor $(10)_b$ is defined according to the conventions of [21]. We

⁵We acknowledge conversations with Kevin Iga who emphasized this point.

next note that the element of the permutation group above obviously satisfies

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 2 \\ 3 \end{bmatrix} \quad (3.2)$$

which implies

$$1 \rightarrow 1, \quad 2 \rightarrow 4, \quad 3 \rightarrow 2, \quad 4 \rightarrow 3, \quad (3.3)$$

and this reveals the cycle (243). We then can write $L_1 = (10)_b(243)$.

Upon applying such considerations to all the L-matrices and the R-matrices, we find

$$\begin{aligned} L_1 &= (10)_b(243), \quad L_2 = (12)_b(123), \quad L_3 = (6)_b(134), \quad L_4 = (0)_b(142), \\ R_1 &= (12)_b(234), \quad R_2 = (9)_b(132), \quad R_3 = (10)_b(143), \quad R_4 = (0)_b(124), \end{aligned} \quad (3.4)$$

for the chiral multiplet and

$$\begin{aligned} L_1 &= (10)_b(1243), \quad L_2 = (12)_b(23), \quad L_3 = (0)_b(14), \quad L_4 = (6)_b(1342), \\ R_1 &= (12)_b(1342), \quad R_2 = (10)_b(23), \quad R_3 = (0)_b(14), \quad R_4 = (13)_b(1243), \end{aligned} \quad (3.5)$$

for the vector multiplet.

It is seen the chiral multiplet is associated with elements of the permutation group that include only three-elements cycles, while the vector multiplet is associated with elements of the permutation group that include only cycles of ‘even length.’ As the formulation we use of the 4D, $\mathcal{N} = 4$ supermultiplet possesses off-shell 4D, $\mathcal{N} = 1$ supersymmetry, the distinction between the vector field and three scalars must be present for one SUSY charge in our subsequent discussion. In fact, we will find that precisely this distinction will be present for all four super charges.

From the adinkra networks shown in Figures # 1 and # 2, the correlations between the link colors, cycles and L-matrices is shown in Table # 1 below.

| | CM | VM |
|--------------|-------|--------|
| <i>BLUE</i> | (243) | (1243) |
| <i>RED</i> | (123) | (23) |
| <i>BLACK</i> | (134) | (14) |
| <i>GREEN</i> | (142) | (1342) |

Table # 1: Adinkra Link Color & Cycles in L-matrices

The work of [21] implies that the information about the even vs. odd length cycles defines a ‘shadow’ of the Hodge duality that respects off-shell 4D, $\mathcal{N} = 1$ supersymmetry.

4 The 4D, $\mathcal{N} = 4$ Vector Supermultiplet with One Off-Shell Supersymmetry

A common treatment of the 4D, $\mathcal{N} = 4$ vector supermultiplet [16, 23, 24] is one where the supersymmetry derivatives are *not* treated symmetrically. In this asymmetrical treatment, one of the supersymmetric covariant derivatives (that can be denoted by D_a) is realized in an off-shell manner, while the remaining three (denoted by $D_a^{\mathcal{I}}$ with $\mathcal{I} = 1, 2$, and 3) are not treated in an off-shell manner.

At the level of component fields, the action for a U(1) 4D, $\mathcal{N} = 4$, supersymmetric vector supermultiplet includes six spin-0 bosons ($A^{\mathcal{I}}$ and $B^{\mathcal{I}}$), one spin-1 gauge boson (A_μ), four spin-1/2 fermions (λ_a and $\psi_a^{\mathcal{I}}$), and seven auxiliary spin-0 fields (d , $F^{\mathcal{I}}$, and $G^{\mathcal{I}}$) and takes the form

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2}(\partial_\mu A^{\mathcal{I}})(\partial^\mu A^{\mathcal{I}}) - \frac{1}{2}(\partial_\mu B^{\mathcal{I}})(\partial^\mu B^{\mathcal{I}}) \\ & + i\frac{1}{2}(\gamma^\mu)^{ab}\psi_a^{\mathcal{I}}\partial_\mu\psi_b^{\mathcal{I}} + \frac{1}{2}(F^{\mathcal{I}})^2 + \frac{1}{2}(G^{\mathcal{I}})^2 \\ & - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\frac{1}{2}(\gamma^\mu)^{cd}\lambda_c\partial_\mu\lambda_d + \frac{1}{2}d^2 \quad . \end{aligned} \quad (4.1)$$

where the gamma matrices throughout our discussion are defined as in Appendix A of [20]. This Lagrangian is invariant up to surface terms with respect to the global supersymmetric transformations define in (2.6) which are here modified to take into account there are now three independent chiral supermultiplets. So we have

$$\begin{aligned} D_a A^{\mathcal{I}} &= \psi_a^{\mathcal{I}} \quad , \\ D_a B^{\mathcal{I}} &= i(\gamma^5)_a{}^b \psi_b^{\mathcal{I}} \quad , \\ D_a \psi_b^{\mathcal{I}} &= i(\gamma^\mu)_{ab} (\partial_\mu A^{\mathcal{I}}) - (\gamma^5 \gamma^\mu)_{ab} (\partial_\mu B^{\mathcal{I}}) \\ &\quad - iC_{ab} F^{\mathcal{I}} + (\gamma^5)_{ab} G^{\mathcal{I}} \quad , \\ D_a F^{\mathcal{I}} &= (\gamma^\mu)_a{}^b (\partial_\mu \psi_b^{\mathcal{I}}) \quad , \\ D_a G^{\mathcal{I}} &= i(\gamma^5 \gamma^\mu)_a{}^b (\partial_\mu \psi_b^{\mathcal{I}}) \quad , \end{aligned} \quad (4.2)$$

under the singlet D-operator acting on the three 4D, $\mathcal{N} = 1$ chiral supermultiplets. For the 4D, $\mathcal{N} = 1$ vector supermultiplet, the realization of the action of the singlet D-operator is given by (2.9) still. In order to realize 4D, $\mathcal{N} = 4$, a triplet $D_a^{\mathcal{I}}$ -operators is required.

There is a well-known realization of the triplet $D_a^{\mathcal{I}}$ -operators that also leaves the action (4.1) invariant up to surface terms. This is provided by

$$\begin{aligned}
D_a^{\mathcal{I}} A^{\mathcal{J}} &= \delta^{\mathcal{I}\mathcal{J}} \lambda_a - \epsilon^{\mathcal{I}\mathcal{J}}{}_{\mathcal{K}} \psi_a^{\mathcal{K}} \quad , \\
D_a^{\mathcal{I}} B^{\mathcal{J}} &= i(\gamma^5)_a{}^b \left[\delta^{\mathcal{I}\mathcal{J}} \lambda_b + \epsilon^{\mathcal{I}\mathcal{J}}{}_{\mathcal{K}} \psi_b^{\mathcal{K}} \right] \quad , \\
D_a^{\mathcal{I}} \psi_b^{\mathcal{J}} &= \delta^{\mathcal{I}\mathcal{J}} \left[i \frac{1}{4} ([\gamma^\mu, \gamma^\nu])_{ab} (\partial_\mu A_\nu - \partial_\nu A_\mu) + (\gamma^5)_{ab} d \right] \\
&\quad + \epsilon^{\mathcal{I}\mathcal{J}}{}_{\mathcal{K}} \left[i(\gamma^\mu)_{ab} (\partial_\mu A^{\mathcal{K}}) + (\gamma^5 \gamma^\mu)_{ab} (\partial_\mu B^{\mathcal{K}}) \right. \\
&\quad \left. - i C_{ab} F^{\mathcal{K}} - (\gamma^5)_{ab} G^{\mathcal{K}} \right] \quad , \\
D_a^{\mathcal{I}} F^{\mathcal{J}} &= (\gamma^\mu)_a{}^b \partial_\mu \left[\delta^{\mathcal{I}\mathcal{J}} \lambda_b - \epsilon^{\mathcal{I}\mathcal{J}}{}_{\mathcal{K}} \psi_b^{\mathcal{K}} \right] \quad , \\
D_a^{\mathcal{I}} G^{\mathcal{J}} &= i(\gamma^5 \gamma^\mu)_a{}^b \partial_\mu \left[-\delta^{\mathcal{I}\mathcal{J}} \lambda_b + \epsilon^{\mathcal{I}\mathcal{J}}{}_{\mathcal{K}} \psi_b^{\mathcal{K}} \right] \quad .
\end{aligned} \tag{4.3}$$

for the component fields of the chiral supermultiplets and for the component fields of the vector supermultiplet we utilize

$$\begin{aligned}
D_a^{\mathcal{I}} A_\mu &= -(\gamma_\mu)_a{}^b \psi_b^{\mathcal{I}} \quad , \\
D_a^{\mathcal{I}} \lambda_b &= i(\gamma^\mu)_{ab} (\partial_\mu A^{\mathcal{I}}) - (\gamma^5 \gamma^\mu)_{ab} (\partial_\mu B^{\mathcal{I}}) \\
&\quad - i C_{ab} F^{\mathcal{I}} - (\gamma^5)_{ab} G^{\mathcal{I}} \quad , \\
D_a^{\mathcal{I}} d &= i(\gamma^5 \gamma^\mu)_a{}^b (\partial_\mu \psi_b^{\mathcal{I}}) \quad .
\end{aligned} \tag{4.4}$$

5 Valise 1D, $N = 16$ Supermultiplet Formulation

In this section we will present the new results of this work. We apply the result in (2.1) and the field re-definitions in (2.14) to the action in (4.1) simultaneously to find

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}(\partial_\tau A^{\mathcal{I}})(\partial_\tau A^{\mathcal{I}}) + \frac{1}{2}(\partial_\tau B^{\mathcal{I}})(\partial_\tau B^{\mathcal{I}}) + \frac{1}{2}(\partial_\tau F^{\mathcal{I}})(\partial_\tau F^{\mathcal{I}}) + \frac{1}{2}(\partial_\tau G^{\mathcal{I}})(\partial_\tau G^{\mathcal{I}}) \\
&\quad + \frac{1}{2}(\partial_\tau A_i)(\partial_\tau A_i) + \frac{1}{2}(\partial_\tau d)(\partial_\tau d) + i \frac{1}{2}(\gamma \cdot \mathcal{T})^{ab} \psi_a^{\mathcal{I}} \partial_\tau \psi_b^{\mathcal{I}} + i \frac{1}{2}(\gamma \cdot \mathcal{T})^{cd} \lambda_c \partial_\tau \lambda_d \quad .
\end{aligned} \tag{5.1}$$

$$\begin{aligned}
D_a A^{\mathcal{J}} &= \psi_a^{\mathcal{J}} \quad , \quad D_a B^{\mathcal{J}} = i(\gamma^5)_a{}^b \psi_b^{\mathcal{J}} \quad , \\
D_a F^{\mathcal{J}} &= (\gamma \cdot \mathcal{T})_a{}^b \psi_b^{\mathcal{J}} \quad , \quad D_a G^{\mathcal{J}} = i(\gamma^5 \gamma \cdot \mathcal{T})_a{}^b \psi_b^{\mathcal{J}} \quad , \\
D_a \psi_b^{\mathcal{J}} &= i(\gamma \cdot \mathcal{T})_{ab} (\partial_\tau A^{\mathcal{J}}) - (\gamma^5 \gamma \cdot \mathcal{T})_{ab} (\partial_\tau B^{\mathcal{J}}) \\
&\quad - i C_{ab} (\partial_\tau F^{\mathcal{J}}) + (\gamma^5)_{ab} (\partial_\tau G^{\mathcal{J}}) \quad ,
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
D_a A_i &= (\gamma_i)_a{}^b \lambda_b \quad , \quad D_a d = i(\gamma^5 \gamma \cdot \mathcal{T})_a{}^b \lambda_b \quad , \\
D_a \lambda_b &= -\frac{i}{2}([\gamma \cdot \mathcal{T}, \gamma^i])_{ab} (\partial_\tau A_i) + (\gamma^5)_{ab} (\partial_\tau d) \quad .
\end{aligned} \tag{5.3}$$

However, for the SU(2) triplet supercovariant derivatives the realization takes the forms

$$\begin{aligned}
D_a^{\mathcal{I}} A^{\mathcal{J}} &= \delta^{\mathcal{I}\mathcal{J}} \lambda_a - \epsilon^{\mathcal{I}\mathcal{J}}{}_{\mathcal{K}} \psi_a^{\mathcal{K}} \quad , \quad D_a^{\mathcal{I}} B^{\mathcal{J}} = i(\gamma^5)_a{}^b \left[\delta^{IJ} \lambda_b + \epsilon^{\mathcal{I}\mathcal{J}}{}_{\mathcal{K}} \psi_b^{\mathcal{K}} \right] \quad , \\
D_a^{\mathcal{I}} F^{\mathcal{J}} &= (\gamma \cdot \mathcal{T})_a{}^b \left[\delta^{\mathcal{I}\mathcal{J}} \lambda_b - \epsilon^{\mathcal{I}\mathcal{J}}{}_{\mathcal{K}} \psi_b^{\mathcal{K}} \right] \quad , \\
D_a^{\mathcal{I}} G^{\mathcal{J}} &= i(\gamma^5 \gamma \cdot \mathcal{T})_a{}^b \left[-\delta^{IJ} \lambda_b + \epsilon^{\mathcal{I}\mathcal{J}}{}_{\mathcal{K}} \psi_b^{\mathcal{K}} \right] \quad , \\
D_a^{\mathcal{I}} \psi_b^{\mathcal{J}} &= \delta^{\mathcal{I}\mathcal{J}} \left[i \frac{1}{2} ([\gamma \cdot \mathcal{T}, \gamma^i])_{ab} (\partial_{\tau} A_i) + (\gamma^5)_{ab} (\partial_{\tau} d) \right] \\
&\quad + \epsilon^{\mathcal{I}\mathcal{J}}{}_{\mathcal{K}} \left[i(\gamma \cdot \mathcal{T})_{ab} (\partial_{\tau} A^{\mathcal{K}}) + (\gamma^5 \gamma \cdot \mathcal{T})_{ab} (\partial_{\tau} B^{\mathcal{K}}) \right. \\
&\quad \left. - i C_{ab} (\partial_{\tau} F^{\mathcal{K}}) - (\gamma^5)_{ab} (\partial_{\tau} G^{\mathcal{K}}) \right] \quad ,
\end{aligned} \tag{5.4}$$

for the fields in the valise adinkra formulation of the three chiral supermultiplets and

$$\begin{aligned}
D_a^{\mathcal{I}} A_i &= -(\gamma_i)_a{}^b \psi_b^{\mathcal{I}} \quad , \quad D_a^{\mathcal{I}} d = i(\gamma^5 \gamma \cdot \mathcal{T})_a{}^b \psi_b^{\mathcal{I}} \quad , \\
D_a^{\mathcal{I}} \lambda_b &= i(\gamma \cdot \mathcal{T})_{ab} (\partial_{\tau} A^{\mathcal{I}}) - (\gamma^5 \gamma \cdot \mathcal{T})_{ab} (\partial_{\tau} B^{\mathcal{I}}) \\
&\quad - i C_{ab} (\partial_{\tau} F^{\mathcal{I}}) - (\gamma^5)_{ab} (\partial_{\tau} G^{\mathcal{I}}) \quad .
\end{aligned} \tag{5.5}$$

for the fields of the valise adinkra formulation of the vector supermultiplet. The equations in this section that involve the D-operators are clearly of the same form as in (2.11) with the indices now ranging as I, J, etc. = 1, ..., 16; i, j, etc. = 1, ..., 16; and \hat{i}, \hat{j} , etc. = 1, ..., 16.

6 Extracting 1D, $N = 16$ Valise Adinkra Matrices

With the results of the previous section in hand, we are now able to extract the L-matrices and R-matrices of the 1D, $N = 16$ adinkra matrices associated with the discussion of the previous chapter. In order to present our results coherently, we use the following notation conventions that are the most obvious appropriate generalizations of (2.11). We now introduce the 1D covariant derivatives $D^{[0]}_{\mathbf{I}}$ and $D^{[\mathcal{I}]}_{\mathbf{I}}$ to act as the holographic images of D_a and $D_a^{\mathcal{I}}$. Their realizations on the valise fields may be expressed in the forms

$$\begin{aligned}
D^{[0]}_{\mathbf{I}} \Phi_i &= i(L_{\mathbf{I}}^{[0]})_{i\hat{k}} \Psi_{\hat{k}} \quad , \quad D^{[0]}_{\mathbf{I}} \Psi_{\hat{k}} = (R_{\mathbf{I}}^{[0]})_{\hat{k}i} (\partial_{\tau} \Phi_i) \quad , \\
D^{[\mathcal{I}]}_{\mathbf{I}} \Phi_i &= i(L_{\mathbf{I}}^{[\mathcal{I}]})_{i\hat{k}} \Psi_{\hat{k}} \quad , \quad D^{[\mathcal{I}]}_{\mathbf{I}} \Psi_{\hat{k}} = (R_{\mathbf{I}}^{[\mathcal{I}]})_{\hat{k}i} (\partial_{\tau} \Phi_i) \quad ,
\end{aligned} \tag{6.1}$$

where above the bosonic and fermionic quantities Φ_i and $\Psi_{\hat{k}}$ respectively take the forms of two sixteen component quantities

$$\begin{aligned}\Phi_i &= \left(A^1, B^1, F^1, G^1, A^2, B^2, F^2, G^2, A^3, B^3, F^3, G^3, \vec{A}, d \right) , \\ \Psi_{\hat{k}} &= -i \left(\psi^1_1, \psi^1_2, \psi^1_3, \psi^1_4, \psi^2_1, \psi^2_2, \psi^2_3, \psi^2_4, \psi^3_1, \psi^3_2, \psi^3_3, \psi^3_4, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \right) ,\end{aligned}\tag{6.2}$$

(where \vec{A} corresponds to the spatial components of the gauge field) as is appropriate in the context of this chapter.

Explicitly we find for the $(L_I^{[0]})_{i\hat{k}}$ matrices

$$\begin{aligned}\left(L_1^{[0]} \right)_{i\hat{k}} &= \begin{bmatrix} (10)_b(243) & 0 & 0 & 0 \\ 0 & (10)_b(243) & 0 & 0 \\ 0 & 0 & (10)_b(243) & 0 \\ 0 & 0 & 0 & (10)_b(1243) \end{bmatrix} , \\ \left(L_2^{[0]} \right)_{i\hat{k}} &= \begin{bmatrix} (12)_b(123) & 0 & 0 & 0 \\ 0 & (12)_b(123) & 0 & 0 \\ 0 & 0 & (12)_b(123) & 0 \\ 0 & 0 & 0 & (4)_b(23) \end{bmatrix} , \\ \left(L_3^{[0]} \right)_{i\hat{k}} &= \begin{bmatrix} (6)_b(134) & 0 & 0 & 0 \\ 0 & (6)_b(134) & 0 & 0 \\ 0 & 0 & (6)_b(134) & 0 \\ 0 & 0 & 0 & (0)_b(14) \end{bmatrix} , \\ \left(L_4^{[0]} \right)_{i\hat{k}} &= \begin{bmatrix} (0)_b(142) & 0 & 0 & 0 \\ 0 & (0)_b(142) & 0 & 0 \\ 0 & 0 & (0)_b(142) & 0 \\ 0 & 0 & 0 & (6)_b(1342) \end{bmatrix} ,\end{aligned}\tag{6.3}$$

and these L-matrices are simply reaffirming relations of colors to eight distinct cycles seen before in chapter three. In a similar manner the $(R_I^{[0]})_{\hat{k}i}$ matrices take the forms

$$\begin{aligned}\left(R_1^{[0]} \right)_{\hat{k}i} &= \begin{bmatrix} (12)_b(234) & 0 & 0 & 0 \\ 0 & (12)_b(234) & 0 & 0 \\ 0 & 0 & (12)_b(234) & 0 \\ 0 & 0 & 0 & (12)_b(1342) \end{bmatrix} , \\ \left(R_2^{[0]} \right)_{\hat{k}i} &= \begin{bmatrix} (9)_b(132) & 0 & 0 & 0 \\ 0 & (9)_b(132) & 0 & 0 \\ 0 & 0 & (9)_b(132) & 0 \\ 0 & 0 & 0 & (10)_b(23) \end{bmatrix} ,\end{aligned}$$

$$\begin{aligned}
\left(R_3^{[0]}\right)_{\hat{k}i} &= \begin{bmatrix} (10)_b(143) & 0 & 0 & 0 \\ 0 & (10)_b(143) & 0 & 0 \\ 0 & 0 & (10)_b(143) & 0 \\ 0 & 0 & 0 & (0)_b(14) \end{bmatrix} \\
\left(R_4^{[0]}\right)_{\hat{k}i} &= \begin{bmatrix} (0)_b(124) & 0 & 0 & 0 \\ 0 & (0)_b(124) & 0 & 0 \\ 0 & 0 & (0)_b(124) & 0 \\ 0 & 0 & 0 & (9)_b(1243) \end{bmatrix}
\end{aligned} \tag{6.4}$$

in the basis defined by (6.2).

This brings us to the explicit results for the triplet L-matrices and R-matrices. By using the same basis we find for the $(L_I^{[0]})_{i\hat{k}}$ matrices

$$\begin{aligned}
\left(L_1^{[1]}\right)_{i\hat{k}} &= \begin{bmatrix} 0 & 0 & 0 & (2)_b(243) \\ 0 & 0 & (15)_b(243) & 0 \\ 0 & (0)_b(243) & 0 & 0 \\ (13)_b(1243) & 0 & 0 & 0 \end{bmatrix}, \\
\left(L_2^{[1]}\right)_{i\hat{k}} &= \begin{bmatrix} 0 & 0 & 0 & (4)_b(123) \\ 0 & 0 & (9)_b(123) & 0 \\ 0 & (6)_b(123) & 0 & 0 \\ (11)_b(23) & 0 & 0 & 0 \end{bmatrix}, \\
\left(L_3^{[1]}\right)_{i\hat{k}} &= \begin{bmatrix} 0 & 0 & 0 & (14)_b(134) \\ 0 & 0 & (3)_b(134) & 0 \\ 0 & (12)_b(134) & 0 & 0 \\ (7)_b(14) & 0 & 0 & 0 \end{bmatrix}, \\
\left(L_4^{[1]}\right)_{i\hat{k}} &= \begin{bmatrix} 0 & 0 & 0 & (8)_b(142) \\ 0 & 0 & (5)_b(142) & 0 \\ 0 & (10)_b(142) & 0 & 0 \\ (1)_b(1342) & 0 & 0 & 0 \end{bmatrix},
\end{aligned} \tag{6.5}$$

with their associated R-matrices given by

$$\left(R_1^{[1]}\right)_{\hat{k}i} = \begin{bmatrix} 0 & 0 & 0 & (7)_b(1342) \\ 0 & 0 & (0)_b(234) & 0 \\ 0 & (15)_b(234) & 0 & 0 \\ (8)_b(234) & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned}
\left(R_2^{[1]}\right)_{\hat{k}i} &= \begin{bmatrix} 0 & 0 & 0 & (13)_b(23) \\ 0 & 0 & (5)_b(132) & 0 \\ 0 & (10)_b(132) & 0 & 0 \\ (1)_b(132) & 0 & 0 & 0 \end{bmatrix} \\
\left(R_3^{[1]}\right)_{\hat{k}i} &= \begin{bmatrix} 0 & 0 & 0 & (14)_b(14) \\ 0 & 0 & (9)_b(143) & 0 \\ 0 & (6)_b(143) & 0 & 0 \\ (11)_b(143) & 0 & 0 & 0 \end{bmatrix} \\
\left(R_4^{[1]}\right)_{\hat{k}i} &= \begin{bmatrix} 0 & 0 & 0 & (4)_b(1243) \\ 0 & 0 & (3)_b(124) & 0 \\ 0 & (12)_b(124) & 0 & 0 \\ (2)_b(124) & 0 & 0 & 0 \end{bmatrix}
\end{aligned} \tag{6.6}$$

We find for the $(L_I^{[2]})_{i\hat{k}}$ matrices

$$\begin{aligned}
\left(L_1^{[2]}\right)_{i\hat{k}} &= \begin{bmatrix} 0 & 0 & (0)_b(243) & 0 \\ 0 & 0 & 0 & (2)_b(243) \\ (15)_b(243) & 0 & 0 & 0 \\ 0 & (13)_b(1234) & 0 & 0 \end{bmatrix}, \\
\left(L_2^{[2]}\right)_{i\hat{k}} &= \begin{bmatrix} 0 & 0 & (6)_b(123) & 0 \\ 0 & 0 & 0 & (4)_b(123) \\ (9)_b(123) & 0 & 0 & 0 \\ 0 & (11)_b(23) & 0 & 0 \end{bmatrix}, \\
\left(L_3^{[2]}\right)_{i\hat{k}} &= \begin{bmatrix} 0 & 0 & (12)_b(134) & 0 \\ 0 & 0 & 0 & (14)_b(134) \\ (3)_b(134) & 0 & 0 & 0 \\ 0 & (7)_b(14) & 0 & 0 \end{bmatrix}, \\
\left(L_4^{[2]}\right)_{i\hat{k}} &= \begin{bmatrix} 0 & 0 & (10)_b(142) & 0 \\ 0 & 0 & 0 & (8)_b(142) \\ (5)_b(142) & 0 & 0 & 0 \\ 0 & (1)_b(1342) & 0 & 0 \end{bmatrix},
\end{aligned} \tag{6.7}$$

with their associated R-matrices given by

$$\left(R_1^{[2]}\right)_{\hat{k}i} = \begin{bmatrix} 0 & 0 & (15)_b(234) & 0 \\ 0 & 0 & 0 & (7)_b(1342) \\ (0)_b(234) & 0 & 0 & 0 \\ 0 & (8)_b(234) & 0 & 0 \end{bmatrix}$$

$$\begin{aligned}
\left(R_2^{[2]}\right)_{\hat{k}i} &= \begin{bmatrix} 0 & 0 & (10)_b(132) & 0 \\ 0 & 0 & 0 & (13)_b(23) \\ (5)_b(132) & 0 & 0 & 0 \\ 0 & (1)_b(132) & 0 & 0 \end{bmatrix} \\
\left(R_3^{[2]}\right)_{\hat{k}i} &= \begin{bmatrix} 0 & 0 & (6)_b(143) & 0 \\ 0 & 0 & 0 & (14)_b(14) \\ (9)_b(143) & 0 & 0 & 0 \\ 0 & (11)_b(143) & 0 & 0 \end{bmatrix} \\
\left(R_4^{[2]}\right)_{\hat{k}i} &= \begin{bmatrix} 0 & 0 & (12)_b(124) & 0 \\ 0 & 0 & 0 & (4)_b(1243) \\ (3)_b(124) & 0 & 0 & 0 \\ 0 & (2)_b(124) & 0 & 0 \end{bmatrix} \tag{6.8}
\end{aligned}$$

We find for the $(L_I^{[3]})_{i\hat{k}}$ matrices

$$\begin{aligned}
\left(L_1^{[3]}\right)_{i\hat{k}} &= \begin{bmatrix} 0 & (15)_b(243) & 0 & 0 \\ (0)_b(243) & 0 & 0 & 0 \\ 0 & 0 & 0 & (2)_b(243) \\ 0 & 0 & (13)_b(1243) & 0 \end{bmatrix}, \\
\left(L_2^{[3]}\right)_{i\hat{k}} &= \begin{bmatrix} 0 & (9)_b(123) & 0 & 0 \\ (6)_b(123) & 0 & 0 & 0 \\ 0 & 0 & 0 & (4)_b(123) \\ 0 & 0 & (11)_b(23) & 0 \end{bmatrix}, \\
\left(L_3^{[3]}\right)_{i\hat{k}} &= \begin{bmatrix} 0 & (3)_b(134) & 0 & 0 \\ (12)_b(134) & 0 & 0 & 0 \\ 0 & 0 & 0 & (14)_b(134) \\ 0 & 0 & (7)_b(14) & 0 \end{bmatrix}, \\
\left(L_4^{[3]}\right)_{i\hat{k}} &= \begin{bmatrix} 0 & (5)_b(142) & 0 & 0 \\ (10)_b(142) & 0 & 0 & 0 \\ 0 & 0 & 0 & (8)_b(142) \\ 0 & 0 & (1)_b(1342) & 0 \end{bmatrix}, \tag{6.9}
\end{aligned}$$

with their associated R-matrices given by

$$\left(R_1^{[3]}\right)_{\hat{k}i} = \begin{bmatrix} 0 & (0)_b(234) & 0 & 0 \\ (15)_b(234) & 0 & 0 & 0 \\ 0 & 0 & 0 & (7)_b(1342) \\ 0 & 0 & (8)_b(234) & 0 \end{bmatrix}$$

$$\begin{aligned}
\left(R_2^{[3]}\right)_{\hat{k}i} &= \begin{bmatrix} 0 & (5)_b(132) & 0 & 0 \\ (10)_b(132) & 0 & 0 & 0 \\ 0 & 0 & 0 & (13)_b(23) \\ 0 & 0 & (1)_b(132) & 0 \end{bmatrix} \\
\left(R_3^{[3]}\right)_{\hat{k}i} &= \begin{bmatrix} 0 & (9)_b(143) & 0 & 0 \\ (6)_b(143) & 0 & 0 & 0 \\ 0 & 0 & 0 & (14)_b(14) \\ 0 & 0 & (11)_b(143) & 0 \end{bmatrix} \\
\left(R_4^{[3]}\right)_{\hat{k}i} &= \begin{bmatrix} 0 & (3)_b(124) & 0 & 0 \\ (12)_b(124) & 0 & 0 & 0 \\ 0 & 0 & 0 & (4)_b(1243) \\ 0 & 0 & (2)_b(124) & 0 \end{bmatrix} \tag{6.10}
\end{aligned}$$

One of the most obvious features about these is that the holographical mechanism for conserving the spin-bundle information of the four dimensional related theory is still present in the 1D, $N = 16$ valise formulation! The explicit way this occurs is by making three observations:

- (a.) examination of the matrices in (6.3) and (6.4) shows a 3:1 ratio of odd-length cycles/even-length cycle within each L-matrix and R-matrix,
- (b.) examination of the matrices in (6.5) - (6.10) continues to show a 3:1 ratio of three odd-length cycles/even-length cycle within each L-matrix and R-matrix cycle, and
- (c.) examination of the matrices in (6.5) - (6.10) shows that only the *same* eight cycles within the permutation group appear within the triplet L-matrices as do appear in the singlet L-matrices.

The first of these observations was expected as it follows from the fact that the first supersymmetry generated by the singlet D_a -operator obviously acts on three 1D, $N = 4$ chiral multiplet adinkras and one 1D, $N = 4$ vector multiplet adinkra.

The second and third observations, however, are striking evidence that the mechanism of using the lengths of cycles of the permutation elements embedded with the L-matrices and R-matrices apparently continues to work as the transformation laws associated with the hologram of the triplet $D_a^{\mathcal{I}}$ -operators possesses the same property and there was *no a priori* reason to expect this conservation of information among all four supercharges. Furthermore, it is seen that in all four sets of L-matrices associated with the supercharges, the same 3:1 ratio of odd-length three-cycles/even cycles is present.

We are thus fortified in our assertion that the 1D, $N = 16$ SUSY quantum mechanical model described by the equations in chapter five constitutes a SUSY hologram of the 4D, $\mathcal{N} = 4$ model described in chapter four.

But is it an off-shell valise adinkra hologram?

7 Previous Results About Off-Shell vs. On-Shell

In order to answer this question, it is useful to both review some previous work [20] that can be used as a foundation upon which an expanded discussion can be built. In this chapter we use the on-shell versus off-shell valise adinkra formulations of the 4D, $\mathcal{N} = 1$ chiral scalar and the vector supermultiplets as our jumping off points.

From the perspective of valise adinkra formulations, L-matrices and R-matrices continue to exist for on-shell theories but with the main distinctions:

- (a.) the i, j , etc. indices have a range according to $1, \dots, d_L$, the \hat{i}, \hat{j} , etc. indices have a range according to $1, \dots, d_R$, where d_L may be distinctly different from d_R , and
- (b.) the algebras for the L-matrices and R-matrices are changed to:

$$\begin{aligned} (L_I)_{i^{\hat{j}}} (R_J)_{\hat{j}}^k + (L_J)_{i^{\hat{j}}} (R_I)_{\hat{j}}^k &= 2 \delta_{IJ} \delta_i^k + (\Delta_{IJ}^L)_i^k, \\ (R_J)_{i^{\hat{j}}} (L_I)_{\hat{j}}^{\hat{k}} + (R_I)_{i^{\hat{j}}} (L_J)_{\hat{j}}^{\hat{k}} &= 2 \delta_{IJ} \delta_i^{\hat{k}} + (\Delta_{IJ}^R)_i^{\hat{k}}. \end{aligned} \quad (7.1)$$

The quantities $(\Delta_{IJ}^L)_i^k$ and $(\Delta_{IJ}^R)_i^{\hat{k}}$ respectively measure the non-closure of the SUSY algebra on the bosons and fermions of the supermultiplet.

7.1 On-Shell Chiral Supermultiplet

The fields (A, B, ψ_a) of the on-shell valise version of the 4D, $\mathcal{N} = 1$ chiral are shown in Figure # 3 below.

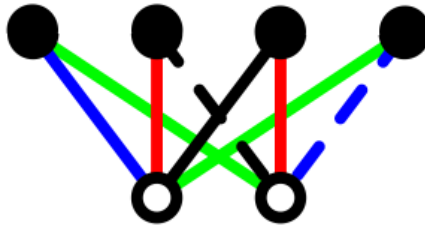


Figure # 3: Adinkra for On-shell Chiral Supermultiplet

which correspond to the D-algebra equations

$$\begin{aligned} D_a A &= \psi_a \quad , \quad D_a B = i(\gamma^5)_a{}^b \psi_b \quad , \\ D_a \psi_b &= i(\gamma \cdot \mathcal{T})_{ab} \partial_\tau A - (\gamma^5 \gamma \cdot \mathcal{T})_{ab} \partial_\tau B \quad . \end{aligned} \quad (7.2)$$

Using (7.2), calculations yield the follow super-commutator algebra

$$\begin{aligned} \{ D_a, D_b \} A &= i 2 (\gamma \cdot \mathcal{T})_{ab} \partial_\tau A \quad , \quad \{ D_a, D_b \} B = i 2 (\gamma \cdot \mathcal{T})_{ab} \partial_\tau B \quad , \\ \{ D_a, D_b \} \psi_c &= i 2 (\gamma \cdot \mathcal{T})_{ab} \partial_\tau \psi_c - i (\gamma^\mu)_{ab} (\gamma_\mu \gamma \cdot \mathcal{T})_c{}^d \partial_\tau \psi_d \quad . \end{aligned} \quad (7.3)$$

The first two of these equations have the expected form for a supersymmetry algebra, but the third term immediately above can be re-expressed as

$$\begin{aligned} \{ D_a, D_b \} \psi_c &= i 2 (\gamma \cdot \mathcal{T})_{ab} \partial_\tau \psi_c + i 2 (\gamma^\mu)_{ab} (\gamma_\mu)_c{}^d \mathcal{K}_d(\psi) \quad , \\ \mathcal{K}_c(\psi) &= - \frac{1}{2} (\gamma \cdot \mathcal{T})_c{}^d \partial_\tau \psi_d \quad , \end{aligned} \quad (7.4)$$

where \mathcal{K}_c measures the ‘non-closure’ of the algebra. From the adinkra in Figure # 3, we define the (2×1) bosonic “field vector” and (4×1) fermionic “field vector”

$$\Phi_i = (A, B) \quad , \quad \Psi_{\hat{k}} = -i(\psi_1, \psi_2, \psi_3, \psi_4) \quad , \quad (7.5)$$

as appropriate for such the adinkra shown in (7.1). This permits us to obtain the following L-matrices and R-matrices.

$$\begin{aligned} (L_1)_{i\hat{k}} &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad , \quad (L_2)_{i\hat{k}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad , \\ (L_3)_{i\hat{k}} &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad , \quad (L_4)_{i\hat{k}} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad , \end{aligned} \quad (7.6)$$

$$\begin{aligned} (R_1)_{i\hat{k}} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \quad , \quad (R_2)_{i\hat{k}} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \quad , \\ (R_3)_{i\hat{k}} &= \begin{bmatrix} 0 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad (R_4)_{i\hat{k}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \quad . \end{aligned} \quad (7.7)$$

Given the matrices in (7.6) and (7.7) we find the following relations hold

$$\begin{aligned} (L_1)_i{}^{\hat{j}} (R_j)_j{}^{\hat{k}} + (L_j)_i{}^{\hat{j}} (R_1)_j{}^{\hat{k}} &= 2 \delta_{ij} \delta_i{}^{\hat{k}} \quad , \\ (R_j)_i{}^{\hat{j}} (L_1)_j{}^{\hat{k}} + (R_1)_i{}^{\hat{j}} (L_j)_j{}^{\hat{k}} &= \delta_{ij} (\mathbf{I})_i{}^{\hat{k}} + [\vec{\alpha} \beta^1]_{ij} \cdot (\vec{\alpha} \beta^1)_i{}^{\hat{k}} \quad . \end{aligned} \quad (7.8)$$

7.2 On-Shell Vector Supermultiplet

In an on-shell vector supermultiplet theory, we have the fields (A_i, λ_a) may be represented by an adinkra of the form

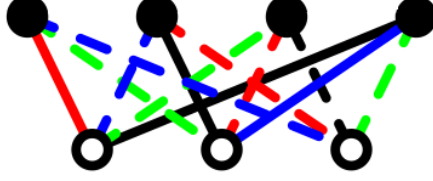


Figure # 4: Adinkra for On-shell Chiral Supermultiplet

which may be chosen to satisfy the equations

$$\begin{aligned} D_a A_i &= (\gamma_i)_a{}^b \lambda_b \quad , \\ D_a \lambda_b &= -i \frac{1}{2} ([\gamma \cdot \mathcal{T}, \gamma^i])_{ab} (\partial_\tau A_i) \quad . \end{aligned} \quad (7.9)$$

From the adinkra above, we define the (3×1) bosonic “field vector” and (4×1) fermionic “field vector”

$$\Phi_i = (A_1, A_2, A_3) \quad , \quad \Psi_{\hat{k}} = -i(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \quad , \quad (7.10)$$

as appropriate for such the adinkra shown in Figure # 4 and once again we calculate the anti-commutator as realized on the remaining fields to find

$$\begin{aligned} \{ D_a, D_b \} A_i &= i 2 (\gamma \cdot \mathcal{T})_{ab} \partial_\tau A_i \quad , \\ \{ D_a, D_b \} \lambda_c &= i 2 (\gamma \cdot \mathcal{T})_{ab} \partial_\tau \lambda_c - i \frac{1}{2} (\gamma^\mu)_{ab} (\gamma_\mu \gamma \cdot \mathcal{T})_c{}^d \partial_\tau \lambda_d \\ &\quad + i \frac{1}{16} ([\gamma^\alpha, \gamma^\beta])_{ab} ([\gamma_\alpha, \gamma_\beta] \gamma \cdot \mathcal{T})_c{}^d \partial_\tau \lambda_d \quad . \end{aligned} \quad (7.11)$$

The final equation of (7.11) shows the presence of *two* non-closure terms. We may rewrite the final line as

$$\begin{aligned} \{ D_a, D_b \} \lambda_c &= i 2 (\gamma \cdot \mathcal{T})_{ab} \partial_\tau \lambda_c + i 2 (\gamma^\mu)_{ab} (\gamma_\mu)_c{}^d \widehat{K}_d(\lambda) \\ &\quad - i \frac{1}{4} ([\gamma^\alpha, \gamma^\beta])_{ab} ([\gamma_\alpha, \gamma_\beta])_c{}^d \widehat{K}_d(\lambda) \quad , \\ \widehat{K}_c(\lambda) &\equiv -\frac{1}{4} (\gamma \cdot \mathcal{T})_c{}^d \partial_\tau \lambda_d \quad , \end{aligned} \quad (7.12)$$

where the non-closure term $\widehat{K}_c(\lambda)$ is introduced. Given the bosonic field vectors, the fermionic field vector, and the valise adinkra in Figure # 4, we find the following L-matrices and R-matrices below.

$$(L_1)_{i\hat{k}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad , \quad (L_2)_{i\hat{k}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad ,$$

$$(\mathbf{L}_3)_{i\hat{k}} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad , \quad (\mathbf{L}_4)_{i\hat{k}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad , \quad (7.13)$$

$$(\mathbf{R}_1)_{\hat{k}i} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \quad , \quad (\mathbf{R}_2)_{\hat{k}i} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad ,$$

$$(\mathbf{R}_3)_{\hat{k}i} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad , \quad (\mathbf{R}_4)_{\hat{k}i} = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad , \quad (7.14)$$

Given the matrices in (7.13) and (7.14) we find the following relations hold

$$\begin{aligned} (\mathbf{L}_1)_{i\hat{j}} (\mathbf{R}_j)^k + (\mathbf{L}_j)_{i\hat{j}} (\mathbf{R}_1)^k &= 2\delta_{1j} \delta_i^k \quad , \\ (\mathbf{R}_j)_{i\hat{j}} (\mathbf{L}_1)^{\hat{k}} + (\mathbf{R}_1)_{i\hat{j}} (\mathbf{L}_j)^{\hat{k}} &= \frac{3}{2} \delta_{1j} (\mathbf{I}_4)_i^{\hat{k}} - \frac{1}{2} [\vec{\alpha}\beta^2]_{1j} \cdot (\vec{\alpha}\beta^2)_i^{\hat{k}} \\ &\quad + \frac{1}{2} [\vec{\alpha}\beta^1]_{1j} \cdot (\vec{\alpha}\beta^1)_i^{\hat{k}} \\ &\quad + \frac{1}{2} [\vec{\alpha}\beta^3]_{1j} \cdot (\vec{\alpha}\beta^3)_i^{\hat{k}} \quad . \end{aligned} \quad (7.15)$$

The results in (7.3), (7.8), (7.11), and (7.15) once more are beautiful examples of SUSY holography. In the cases of (7.3), and (7.11) the SUSY commutator algebra closes on the bosons. In the corresponding algebra of the L-matrices and R-matrices, the quantity $(\Delta_{1j}^L)_i^k$ is identically zero.

However, a careful examination of $(\Delta_{1j}^R)_i^{\hat{k}}$ in each of the respective cases (7.8) and (7.15) reveals an even more striking exhibition of SUSY holography.

In the case of (7.4), if we look at the non-closure terms, it is seen that there appear four linearly independent matrices $(\gamma^\mu)_{ab}$ on the right hand side of the equation. In a similar manner, if we look at $(\Delta_{1j}^R)_i^{\hat{k}}$ (as defined by the second line of (7.8)), we see there appear precisely four independent matrices δ_{1j} , and $(\vec{\alpha}\beta^1)_{1j}$.

In the case of (7.12), if we look at the non-closure terms, it is seen that there appear ten linearly independent matrices $(\gamma^\mu)_{ab}$, and $([\gamma_\alpha, \gamma_\beta])_{ab}$ on the right hand side of the equation. In a similar manner, if we look at $(\Delta_{1j}^R)_i^{\hat{k}}$ (as defined by the second line of (7.15)), we see there appear precisely ten independent matrices δ_{1j} , $(\vec{\alpha}\beta^2)_{1j}$, $(\vec{\alpha}\beta^1)_{1j}$, $(\vec{\alpha}\beta^3)_{1j}$ on the right hand side of the equation.

Finally, by looking at the results in (7.4), and (7.12), it is seen that the concept of an “off-shell central charge” collapses for one dimensional SUSY theories! In higher dimensions imposing the conditions $\mathcal{K}_c(\psi) = 0$ or $\widehat{K}_c(\lambda) = 0$ is equivalent to imposition of a equation of motion restrictions on fermions. However, for a one dimensional SUSY theory, this is also equivalent to demanding that all fermionic fields are constants and by consistency all bosonic field can be at most linear functions of τ .

7.3 The 0-Brane Reduced Algebra

Here we present the form of the super-commutator algebra of the supercovariant derivatives whose realizations are given in 5.2 - 5.5.

$$\begin{aligned}
\{D_a^{\mathcal{I}}, D_b^{\mathcal{J}}\} A^K &= i 2 \delta^{\mathcal{I}\mathcal{J}} (\gamma \cdot \mathcal{T})_{ab} \partial_\tau A^K - 2 \epsilon^{\mathcal{I}\mathcal{J}\mathcal{K}} (\gamma^5)_{ab} \partial_\tau d \\
&\quad - 2 \kappa^{\mathcal{I}\mathcal{J}\mathcal{K}\mathcal{M}} [i C_{ab} \partial_\tau F^{\mathcal{M}} + (\gamma^5)_{ab} \partial_\tau G^{\mathcal{M}}] \quad , \\
\{D_a^{\mathcal{I}}, D_b^{\mathcal{J}}\} B^K &= i 2 \delta^{\mathcal{I}\mathcal{J}} (\gamma \cdot \mathcal{T})_{ab} \partial_\tau B^K + 2 i \epsilon^{\mathcal{I}\mathcal{J}\mathcal{K}} C_{ab} \partial_\tau d \quad , \\
\{D_a^{\mathcal{I}}, D_b^{\mathcal{J}}\} F^K &= i 2 \delta^{\mathcal{I}\mathcal{J}} (\gamma \cdot \mathcal{T})_{ab} \partial_\tau F^K + 2 \epsilon^{\mathcal{I}\mathcal{J}\mathcal{K}} (\gamma^5 \gamma \cdot \mathcal{T})_{ab} \partial_\tau d \\
&\quad + 2 \kappa^{\mathcal{I}\mathcal{J}\mathcal{K}\mathcal{M}} [-i C_{ab} \partial_\tau A^{\mathcal{M}} + (\gamma^5 \gamma \cdot \mathcal{T})_{ab} \partial_\tau G^{\mathcal{M}}] \quad , \\
\{D_a^{\mathcal{I}}, D_b^{\mathcal{J}}\} G^K &= i 2 \delta^{\mathcal{I}\mathcal{J}} (\gamma \cdot \mathcal{T})_{ab} \partial_\tau G^K + 2 \epsilon^{\mathcal{I}\mathcal{J}\mathcal{K}} (\gamma^5 \gamma^i)_{ab} \partial_\tau A_i \\
&\quad - 2 \kappa^{\mathcal{I}\mathcal{J}\mathcal{K}\mathcal{M}} [(\gamma^5)_{ab} \partial_\tau A^{\mathcal{M}} + (\gamma^5 \gamma \cdot \mathcal{T})_{ab} \partial_\tau F^{\mathcal{M}}] \quad , \\
\{D_a^{\mathcal{I}}, D_b^{\mathcal{J}}\} d &= i 2 \delta^{\mathcal{I}\mathcal{J}} (\gamma \cdot \mathcal{T})_{ab} \partial_\tau d \\
&\quad + 2 \epsilon^{\mathcal{I}\mathcal{J}\mathcal{K}} [(\gamma^5)_{ab} \partial_\tau A^K - i C_{ab} \partial_\tau B^K + (\gamma^5 \gamma \cdot \mathcal{T})_{ab} \partial_\tau F^K] \quad , \\
\{D_a^{\mathcal{I}}, D_b^{\mathcal{J}}\} A_i &= i 2 \delta^{\mathcal{I}\mathcal{J}} (\gamma \cdot \mathcal{T})_{ab} \partial_\tau A_i - 2 \epsilon^{\mathcal{I}\mathcal{J}\mathcal{K}} (\gamma^5 \gamma_i)_{ab} \partial_\tau G^K \quad , \\
\{D_a^{\mathcal{I}}, D_b^{\mathcal{J}}\} \lambda_c &= i 2 \delta^{\mathcal{I}\mathcal{J}} (\gamma \cdot \mathcal{T})_{ab} \partial_\tau \lambda_c - i \epsilon^{\mathcal{I}\mathcal{J}\mathcal{K}} [C_{ab} (\gamma \cdot \mathcal{T})_c^d - (\gamma^5)_{ab} (\gamma^5 \gamma \cdot \mathcal{T})_c^d \\
&\quad - (\gamma^5 \gamma^\nu)_{ab} (\gamma^5 \gamma_\nu \gamma \cdot \mathcal{T})_c^d] \partial_\tau \psi_d^K \quad , \\
\{D_a^{\mathcal{I}}, D_b^{\mathcal{J}}\} \psi_c^K &= i 2 \delta^{\mathcal{I}\mathcal{J}} (\gamma \cdot \mathcal{T})_{ab} \partial_\tau \psi_c^K + i \epsilon^{\mathcal{I}\mathcal{J}\mathcal{K}} [C_{ab} (\gamma \cdot \mathcal{T})_c^d - (\gamma^5)_{ab} (\gamma^5 \gamma \cdot \mathcal{T})_c^d \\
&\quad - (\gamma^5 \gamma^\nu)_{ab} (\gamma^5 \gamma_\nu \gamma \cdot \mathcal{T})_c^d] \partial_\tau \lambda_d \\
&\quad - i \kappa^{\mathcal{I}\mathcal{J}\mathcal{K}\mathcal{M}} [C_{ab} (\gamma \cdot \mathcal{T})_c^d + (\gamma^5)_{ab} (\gamma^5 \gamma \cdot \mathcal{T})_c^d \\
&\quad + (\gamma^5 \gamma^\nu)_{ab} (\gamma^5 \gamma_\nu \gamma \cdot \mathcal{T})_c^d] \partial_\tau \psi_d^{\mathcal{M}} \quad .
\end{aligned} \tag{7.16}$$

where the quantity is defined by $\kappa^{\mathcal{I}\mathcal{J}\mathcal{K}\mathcal{M}} \equiv \delta^{IM} \delta^{JK} - \delta^{IK} \delta^{JM}$. We should mention that these were obtained by applying the 0-brane reduction procedure to the the results presented in [19].

The super-commutator for the singlet-supercovariant D_a with the triplet-supercovariant $D_a^{\mathcal{I}}$ takes the form

$$\begin{aligned}
\{D_a, D_b^{\mathcal{I}}\} A^{\mathcal{J}} &= i 2 \epsilon^{\mathcal{I} \mathcal{J} \kappa} C_{ab} \partial_{\tau} F^{\kappa} \\
\{D_a, D_b^{\mathcal{I}}\} B^{\mathcal{J}} &= i 2 \epsilon^{\mathcal{I} \mathcal{J} \kappa} C_{ab} \partial_{\tau} G^{\kappa} \\
\{D_a, D_b^{\mathcal{I}}\} F^{\mathcal{J}} &= i 2 \epsilon^{\mathcal{I} \mathcal{J} \kappa} C_{ab} \partial_{\tau} A^{\kappa} \\
\{D_a, D_b^{\mathcal{I}}\} G^{\mathcal{J}} &= i 2 \epsilon^{\mathcal{I} \mathcal{J} \kappa} C_{ab} \partial_{\tau} B^{\kappa} \\
\{D_a, D_b^{\mathcal{I}}\} \psi_c^{\mathcal{J}} &= i 2 \epsilon^{\mathcal{I} \mathcal{J} \kappa} C_{ab} (\gamma \cdot \mathcal{T})_c^d \partial_{\tau} \psi_d^{\kappa} \\
\{D_a, D_b^{\mathcal{I}}\} d &= 0 \\
\{D_a, D_b^{\mathcal{I}}\} \vec{A} &= 0 \\
\{D_a, D_b^{\mathcal{I}}\} \lambda_c &= 0 \quad .
\end{aligned} \tag{7.17}$$

The first five of these equations inform us that there is a triplet central charge $\mathcal{Z}^{\mathcal{I}}$ that appears algebraically as

$$\{D_a, D_b^{\mathcal{I}}\} = i 2 C_{ab} \mathcal{Z}^{\mathcal{I}} \tag{7.18}$$

and that acts on the fields of the chiral multiplet according to:

$$\begin{aligned}
\mathcal{Z}^{\mathcal{I}} (A^{\mathcal{J}}) &= \epsilon^{\mathcal{I} \mathcal{J} \kappa} \partial_{\tau} F^{\kappa} \quad , \quad \mathcal{Z}^{\mathcal{I}} (B^{\mathcal{J}}) = \epsilon^{\mathcal{I} \mathcal{J} \kappa} \partial_{\tau} G^{\kappa} \quad , \\
\mathcal{Z}^{\mathcal{I}} (F^{\mathcal{J}}) &= \epsilon^{\mathcal{I} \mathcal{J} \kappa} \partial_{\tau} A^{\kappa} \quad , \quad \mathcal{Z}^{\mathcal{I}} (G^{\mathcal{J}}) = \epsilon^{\mathcal{I} \mathcal{J} \kappa} \partial_{\tau} B^{\kappa} \quad , \\
\mathcal{Z}^{\mathcal{I}} (\psi_a^{\mathcal{J}}) &= \epsilon^{\mathcal{I} \mathcal{J} \kappa} (\gamma \cdot \mathcal{T})_c^d \partial_{\tau} \psi_d^{\kappa} \quad .
\end{aligned} \tag{7.19}$$

This same triplet central charge acts very differently on the fields of the vector supermultiplet where we see

$$\mathcal{Z}^{\mathcal{I}} (\lambda_a) = \mathcal{Z}^{\mathcal{I}} (\vec{A}) = \mathcal{Z}^{\mathcal{I}} (d) = 0 \quad . \tag{7.20}$$

So based on the experience of previous examinations based on valise adinkras, the 1D, L-matrices and R-matrices derived in chapter six do *not* correspond to an off-shell valise.

8 Conclusion, Summary & Prospectus

In the work, we have reached a milestone of establishing a 1D, $N = 16$ formulation of the 4D, $\mathcal{N} = 4$ abelian vector supermultiplet realized as a valise. At the level of

representation theory, the formulation is expected to capture faithfully properties of the four dimensional theory.

We have explicitly seen that the distinction between 4D, $\mathcal{N} = 1$ chiral and 4D, $\mathcal{N} = 1$ vector supermultiplets as characterized by the length of cycles in related L-matrices and R-matrices is retained even though only one of four SUSY charges is realized in an off-shell manner.

This work establishes a new platform from which to explore a very old problem, “Do there exist a set of fields that contain those of the on-shell 4D, $\mathcal{N} = 4$ abelian vector supermultiplet as a subset and allow for the off-shell realization of four spacetime supercharges?”

There is a widely held view that the answer to this question is negative. One of the most cited reason given for supporting this viewpoint is a ‘no-go theorem’ [25]. We do not disagree with this result. However, we have long asserted that one of the assumptions at its foundation is *a priori* about dynamics. In particular, the authors observe:

Since all spinor auxiliary fields come in pairs (one as the Lagrange multiplier of the other), the total Fermi dimensionality of the off-shell representation is thus determined modulo 2d ($=8N$) by the total dimensionality of the physical Fermi fields.

If this assumption is dropped, would the result of the no-go theorem change? This possibility is held out even in this work itself. It is in this domain of relaxing this assumption we wish to probe using adinkra-enabled methodology.

We have arrived at a rather precise reformulation of the off-shell problem strictly in terms of linear algebra. The problem is to find the smallest integer p for which sixteen distinct $(16 + 4p) \times (16 + 4p)$ L-matrices can be constructed that satisfy two conditions:

- (a.) they must contain the 16×16 L-matrix sub-blocks of (6.3) - (6.10), and
- (b.) realize the “Garden Algebra” conditions” in (2.11) - (2.13).

We strongly suspect the result for the no-go theorem will change, at least at the level of valise adinkras, if the assumption mentioned above is dropped. Superspace methods imply that there is necessarily a solution for $p = 131,068$, but the challenge is to find smaller solutions.

Firstly, as seen in our discussion of the 4D, $\mathcal{N} = 1$ chiral and vector supermultiplets, the problem of auxiliary fields is equivalent to adding more nodes to the

adinkra corresponding to the on-shell theory. In the example of the 4D, $\mathcal{N} = 1$ supermultiplets, the addition of such nodes corresponds to enlarging the L-matrices and R-matrices (analogous to starting with those of 7.6, 7.7, 7.13, 7.14) in such a way as to satisfy the conditions in (2.11), (2.12), and (2.13).

This is a problem that has similarities to a problem in cryptography and is familiar to anyone who has seen the television game show, “Wheel of Fortune” or in some versions of a crossword puzzle. Some letters, but not all, in words are given, and the point is reconstruct the complete words. The “Garden Algebras” in this context serves as the dictionary of acceptable words.

In our 1D, $N = 4$ examples, the matrices of 7.6, 7.7, 7.13, and 7.14 act as the initial letters and the complete matrices (given in 2.17 - 2.20) satisfying the conditions in (2.12) - (2.13) play the role of the completed words. Apparently for the 1D, $N = 16$ supermultiplet, the matrices of chapter six play the role of the initial letters, and the quest should be to complete them so that their augmentation into even larger square arrays that satisfy the conditions in (2.12) - (2.13).

Even if there is a solution for the valise adinkra, there would remain the problem of lifting of the nodes and the restoration of 4D Lorentz invariance. So success at the level of adinkras is no guarantee for the full field theory. However, even in this case, the nature of any obstruction would be clarified.

In future works, we will report on our continuing efforts.

*“The world will never be the same once you’ve seen
it through the eyes of Forrest Gump.” - Forrest Gump*

*“Everything should be made as simple as possible, but not
simpler.” - Attributed to A. Einstein*

Added Note In Proof

During the course of this work, it became apparent that the color assignments to two of the adinkra graphs in the work of [20] are inconsistent with the rest of that work. To obtain a consistent assignment, whenever the vector supermultiplet adinkras are illustrated there, the follow color reassignments need to be made:

- (a.) orange links \rightarrow purple links,
- (b.) green links \rightarrow orange links, and
- (c.) purple links \rightarrow green links.

In our current work this inconsistency has been eliminated.

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